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# Introduction

This volume is a result of the conference on higher local fields in Münster, August 29–September 5, 1999, which was supported by SFB 478 “Geometrische Strukturen in der Mathematik”. The conference was organized by I. Fesenko and F. Lorenz. We gratefully acknowledge great hospitality and tremendous efforts of Falko Lorenz which made the conference vibrant.

Class field theory as developed in the first half of this century is a fruitful generalization and extension of Gauss reciprocity law; it describes abelian extensions of number fields in terms of objects associated to these fields. Since its construction, one of the important themes of number theory was its generalizations to other classes of fields or to non-abelian extensions.

In modern number theory one encounters very naturally schemes of finite type over $\mathbb{Z}$. A very interesting direction of generalization of class field theory is to develop a theory for higher dimensional fields — finitely generated fields over their prime subfields (or schemes of finite type over $\mathbb{Z}$ in the geometric language). Work in this subject, higher (dimensional) class field theory, was initiated by A.N. Parshin and K. Kato independently about twenty five years ago. For an introduction into several global aspects of the theory see W. Raskind’s review on abelian class field theory of arithmetic schemes.

One of the first ideas in higher class field theory is to work with the Milnor $K$-groups instead of the multiplicative group in the classical theory. It is one of the principles of class field theory for number fields to construct the reciprocity map by some blending of class field theories for local fields. Somewhat similarly, higher dimensional class field theory is obtained as a blending of higher dimensional *local* class field theories, which treat abelian extensions of *higher local fields*. In this way, the higher local fields were introduced in mathematics.

A precise definition of higher local fields will be given in section 1 of Part I; here we give an example. A complete discrete valuation field $K$ whose residue field is isomorphic to a usual local field with finite residue field is called a two-dimensional local field. For example, fields $\mathbb{F}_p((T))((S))$, $\mathbb{Q}_p((S))$ and

$$\mathbb{Q}_p\{\{T\}\} = \left\{ \sum_{-\infty}^{+\infty} a_i T^i : a_i \in \mathbb{Q}_p, \inf v_p(a_i) > -\infty, \lim_{i\to-\infty} v_p(a_i) = +\infty \right\}$$

($v_p$ is the $p$-adic valuation map) are two-dimensional local fields. Whereas the first two fields above can be viewed as generalizations of functional local fields, the latter field comes in sight as an arithmetical generalization of $\mathbb{Q}_p$.

In the classical local case, where $K$ is a complete discrete valuation field with finite residue field, the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ of the maximal abelian extension of $K$ is approximated by the multiplicative group $K^*$; and the reciprocity map

$$K^* \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

is close to an isomorphism (it induces an isomorphism between the group $K^*/N_{L/K}L^*$ and $\mathrm{Gal}(L/K)$ for a finite abelian extension $L/K$, and it is injective with everywhere dense image). For two-dimensional local fields $K$ as above, instead of the multiplicative group $K^*$, the Milnor $K$-group $K_2(K)$ (cf. Some Conventions and section 2 of Part I) plays an important role. For these fields there is a reciprocity map

$$K_2(K) \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

which is approximately an isomorphism (it induces an isomorphism between the group $K_2(K)/N_{L/K}K_2(L)$ and $\mathrm{Gal}(L/K)$ for a finite abelian extension $L/K$, and it has everywhere dense image; but it is not injective: the quotient of $K_2(K)$ by the kernel of the reciprocity map can be described in terms of topological generators, see section 6 Part I).

Similar statements hold in the general case of an $n$-dimensional local field where one works with the Milnor $K_n$-groups and their quotients (sections 5,10,11 of Part I); and even class field theory of more general classes of complete discrete valuation fields can be reasonably developed (sections 13,16 of Part I).

Since $K_1(K) = K^*$, higher local class field theory contains the classical local class field theory as its one-dimensional version.

The aim of this book is to provide an introduction to higher local fields and render the main ideas of this theory. The book grew as an extended version of talks given at the conference in Münster. Its expository style aims to introduce the reader into the subject and explain main ideas, methods and constructions (sometimes omitting details). The contributors applied essential efforts to explain the most important features of their subjects.

Hilbert's words in Zahlbericht that precious treasures are still hidden in the theory of abelian extensions are still up-to-date. We hope that this volume, as the first collection of main strands of higher local field theory, will be useful as an introduction and guide on the subject.

**The first part** presents the theory of higher local fields, very often in the more general setting of complete discrete valuation fields.

Section 1, written by I. Zhukov, introduces higher local fields and topologies on their additive and multiplicative groups. Subsection 1.1 contains all basic definitions and is referred to in many other sections of the volume. The topologies are defined in such a

way that the topology of the residue field is taken into account; the price one pays is that multiplication is not continuous in general, however it is sequentially continuous which allows one to expand elements into convergent power series or products.

Section 2, written by O. Izhboldin, is a short review of the Milnor $K$-groups and Galois cohomology groups. It discusses $p$-torsion and cotorsion of the groups $K_n(F)$ and $K_n^t(F) = K_n(F)/\cap_{l\geqslant 1} lK_n(F)$, an analogue of Satz 90 for the groups $K_n(F)$ and $K_n^t(F)$, and computation of $H_m^{n+1}(F)$ where $F$ is either the rational function field in one variable $F = k(t)$ or the formal power series $F = k((t))$.

Appendix to Section 2, written by M. Kurihara and I. Fesenko, contains some basic definitions and properties of differential forms and Kato's cohomology groups in characteristic $p$ and a sketch of the proof of Bloch–Kato–Gabber's theorem which describes the differential symbol from the Milnor $K$-group $K_n(F)/p$ of a field $F$ of positive characteristic $p$ to the differential module $\Omega_F^n$.

Section 4, written by J. Nakamura, presents main steps of the proof of Bloch–Kato's theorem which states that the norm residue homomorphism

$$K_q(K)/m \to H^q(K, \mathbb{Z}/m\,(q))$$

is an isomorphism for a henselian discrete valuation field $K$ of characteristic 0 with residue field of positive characteristic. This theorem and its proof allows one to simplify Kato's original approach to higher local class field theory.

Section 5, written by M. Kurihara, is a presentation of main ingredients of Kato's higher local class field theory.

Section 6, written by I. Fesenko, is concerned with certain topologies on the Milnor $K$-groups of higher local fields $K$ which are related to the topology on the multiplicative group; their properties are discussed and the structure of the quotient of the Milnor $K$-groups modulo the intersection of all neighbourhoods of zero is described. The latter quotient is called a topological Milnor $K$-group; it was first introduced by Parshin.

Section 7, written by I. Fesenko, describes Parshin's higher local class field theory in characteristic $p$, which is relatively easy in comparison with the cohomological approach.

Section 8, written by S. Vostokov, is a review of known approaches to explicit formulas for the (wild) Hilbert symbol not only in the one-dimensional case but in the higher dimensional case as well. One of them, Vostokov's explicit formula, is of importance for the study of topological Milnor $K$-groups in section 6 and the existence theorem in section 10.

Section 9, written by M. Kurihara, introduces his exponential homomorphism for a complete discrete valuation field of characteristic zero, which relates differential forms and the Milnor $K$-groups of the field, thus helping one to get an additional information on the structure of the latter. An application to explicit formulas is discussed in subsection 9.2.

Section 10, written by I. Fesenko, presents his explicit method to construct higher local class field theory by using topological $K$-groups and a generalization of Neukirch–

Hazewinkel's axiomatic approaches to class field theory. Subsection 10.2 presents another simple approach to class field theory in the characteristic $p$ case. The case of characteristic 0 is sketched using a concept of Artin–Schreir trees of extensions (as those extensions in characteristic 0 which are twinkles of the characteristic $p$ world). The existence theorem is discussed in subsection 10.5, being built upon the results of sections 6 and 8.

Section 11, written by M. Spieß, provides a glimpse of Koya's and his approach to the higher local reciprocity map as a generalization of the classical class formations approach to the level of complexes of Galois modules.

Section 12, written by M. Kurihara, sketches his classification of complete discrete valuation fields $K$ of characteristic 0 with residue field of characteristic $p$ into two classes depending on the behaviour of the torsion part of a differential module. For each of these classes, subsection 12.1 characterizes the quotient filtration of the Milnor $K$-groups of $K$, for all sufficiently large members of the filtration, as a quotient of differential modules. For a higher local field the previous result and higher local class field theory imply certain restrictions on types of cyclic extensions of the field of sufficiently large degree. This is described in 12.2.

Section 13, written by M. Kurihara, describes his theory of cyclic $p$-extensions of an absolutely unramified complete discrete valuation field $K$ with *arbitrary* residue field of characteristic $p$. In this theory a homomorphism is constructed from the $p$-part of the group of characters of $K$ to Witt vectors over its residue field. This homomorphism satisfies some important properties listed in the section.

Section 14, written by I. Zhukov, presents some explicit methods of constructing abelian extensions of complete discrete valuation fields. His approach to explicit equations of a cyclic extension of degree $p^n$ which contains a given cyclic extension of degree $p$ is explained. An application to the structure of topological $K$-groups of an absolutely unramified higher local field is given in subsection 14.6.

Section 15, written by J. Nakamura, contains a list of all known results on the quotient filtration on the Milnor $K$-groups (in terms of differential forms of the residue field) of a complete discrete valuation field. It discusses his recent study of the case of a tamely ramified field of characteristic 0 with residue field of characteristic $p$ by using the exponential map of section 9 and a syntomic complex.

Section 16, written by I. Fesenko, is devoted to his generalization of one-dimensional class field theory to a description of abelian totally ramified $p$-extensions of a complete discrete valuation field with arbitrary non separably-$p$-closed residue field. In particular, subsection 16.3 shows that two such extensions coincide if and only if their norm groups coincide. An illustration to the theory of section 13 is given in subsection 16.4.

Section 17, written by I. Zhukov, is a review of his recent approach to ramification theory of a complete discrete valuation field with residue field whose $p$-basis consists of at most one element. One of important ingredients of the theory is Epp's theorem on elimination of wild ramification (subsection 17.1). New lower and upper filtrations are defined (so that cyclic extensions of degree $p$ may have non-integer ramification breaks,

see examples in subsection 17.2). One of the advantages of this theory is its compatibility with the reciprocity map. A refinement of the filtration for two-dimensional local fields which is compatible with the reciprocity map is discussed.

Section 18, written by L. Spriano, presents ramification theory of monogenic extensions of complete discrete valuation fields; his recent study demonstrates that in this case there is a satisfactory theory if one systematically uses a generalization of the function $i$ and not $s$ (see subsection 18.0 for definitions). Relations to Kato's conductor are discussed in 18.2 and 18.3.

These sections 17 and 18 can be viewed as the rudiments of higher ramification theory; there are several other approaches. Still, there is no satisfactory general ramification theory for complete discrete valuation fields in the imperfect residue field case; to construct such a theory is a challenging problem.

Without attempting to list all links between the sections we just mention several paths (2 means Section 2 and Appendix to Section 2)

| | |
|---|---|
| $1 \to 6 \to 7$ | (leading to Parshin's approach in positive characteristic), |
| $2 \to 4 \to 5 \to 11$ | (leading to Kato's cohomological description |
| | of the reciprocity map and generalized class formations), |
| $8.3 \to 6 \to 10$ | (explicit construction of the reciprocity map), |
| $5 \to 12 \to 13 \to 15$, | (structure of the Milnor $K$-groups of the fields |
| $1 \to 10 \to 14, 16$ | and more explicit study of abelian extensions), |
| $8, 9$ | (explicit formulas for the Hilbert norm symbol |
| | and its generalizations), |
| $1 \to 10 \to 17, 18$ | (aspects of higher ramification theory). |

A special place in this volume (between Part I and Part II) is occupied by the work of K. Kato on the existence theorem in higher local class field theory which was produced in 1980 as an IHES preprint and has never been published. We are grateful to K. Kato for his permission to include this work in the volume. In it, viewing higher local fields as ring objects in the category of iterated pro-ind-objects, a definition of open subgroups in the Milnor $K$-groups of the fields is given. The self-duality of the additive group of a higher local field is proved. By studying norm groups of cohomological objects and using cohomological approach to higher local class field theory the existence theorem is proved. An alternative approach to the description of norm subgroups of Galois extensions of higher local fields and the existence theorem is contained in sections 6 and 10.

**The second part** is concerned with various applications and connections of higher local fields with several other areas.

Section 1, written by A.N. Parshin, describes some first steps in extending Tate–Iwasawa's analytic method to define an $L$-function in higher dimensions; historically the latter problem was one of the stimuli of the work on higher class field theory. For generalizing this method the author advocates the usefulness of the classical Riemann–Hecke approach (subsection 1.1), his adelic complexes (subsection 1.2.2) together with his generalization of Krichever's correspondence (subsection 1.2.1). He analyzes dimension 1 types of functions in subsection 1.3 and discusses properties of the lattice of commensurable classes of subspaces in the adelic space associated to a divisor on an algebraic surface in subsection 1.4.

Section 2, written by D. Osipov, is a review of his recent work on adelic constructions of direct images of differentials and symbols in the two-dimensional case in the relative situation. In particular, reciprocity laws for relative residues of differentials and symbols are introduced and applied to a construction of the Gysin map for Chow groups.

Section 3, written by A.N. Parshin, presents his theory of Bruhat–Tits buildings over higher dimensional local fields. The theory is illustrated with the buildings for $PGL(2)$ and $PGL(3)$ for one- and two-dimensional local fields.

Section 4, written by E.-U. Gekeler, provides a survey of relations between Drinfeld modules and higher dimensional fields of positive characteristic.

Section 5, written by M. Kapranov, sketches his recent approach to elements of harmonic analysis on algebraic groups over functional two-dimensional local fields. For a two-dimensional local field subsection 5.4 introduces a Hecke algebra which is formed by operators which integrate pro-locally-constant complex functions over a non-compact domain.

Section 6, written by L. Herr, is a survey of his recent study of applications of Fontaine's theory of $p$-adic representations of local fields ($\Phi-\Gamma$-modules) to Galois cohomology of local fields and explicit formulas for the Hilbert symbol (subsections 6.4–6.6). The two Greek letters lead to two-dimensional local objects (like $\mathcal{O}_{\mathcal{E}(K)}$ introduced in subsection 6.3).

Section 7, written by I. Efrat, introduces recent advances in the zero-dimensional anabelian geometry, that is a characterization of fields by means of their absolute Galois group (for finitely generated fields and for higher local fields). His method of construction of henselian valuations on fields which satisfy some $K$-theoretical properties is presented in subsection 10.3, and applications to an algebraic proof of the local correspondence part of Pop's theorem and to higher local fields are given.

Section 8, written by A. Zheglov, presents his study of two dimensional local skew fields which was initiated by A.N. Parshin. If the skew field has one-dimensional residue field which is in its centre, then one is naturally led to the study of automorphisms of the residue field which are associated to a local parameter of the skew field. Results on such automorphisms are described in subsections 8.2 and 8.3.

Section 9, written by I. Fesenko, is an exposition of his recent work on noncommutative local reciprocity maps for totally ramified Galois extensions with arithmetically

profinite group (for instance $p$-adic Lie extensions). These maps in general are not homomorphisms but Galois cycles; a description of their image and kernel is included.

Section 10, written by B. Erez, is a concise survey of Galois module theory links with class field theory; it lists several open problems.

The theory of higher local fields has several interesting aspects and applications which are not contained in this volume. One of them is the work of Kato on applications of an explicit formula for the reciprocity map in higher local fields to calculations of special values of the $L$-function of a modular form. There is some interest in two-dimensional local fields (especially of the functional type) in certain parts of mathematical physics, infinite group theory and topology where formal power series objects play a central role.

Prerequisites for most sections in the first part of the book are small: local fields and local class field theory, for instance, as presented in Serre's "Local Fields", Iwasawa's "Local Class Field Theory" or Fesenko–Vostokov's "Local Fields and Their Extensions" (the first source contains a cohomological approach whereas the last two are cohomology free) and some basic knowledge of Milnor $K$-theory of discrete valuation fields (for instance Chapter IX of the latter book). See also Some Conventions and Appendix to Section 2 of Part I where we explain several notions useful for reading Part I.

We thank P. Schneider for his support of the conference and work on this volume. The volume is typed using a modified version of osudeG style (written by Walter Neumann and Larry Siebenmann and available from the public domain of Department of Mathematics of Ohio State University, pub/osutex); thanks are due to Larry for his advice on aspects of this style and to both Walter and Larry for permission to use it.

Ivan Fesenko  Masato Kurihara  September 2000

## Contents

## Some Conventions

The notation $X \subset Y$ means that $X$ is a subset of $Y$.

For an abelian group $A$ written additively denote by $A/m$ the quotient group $A/mA$ where $mA = \{ma : a \in A\}$ and by ${}_mA$ the subgroup of elements of order dividing $m$. The subgroup of torsion elements of $A$ is denoted by $\operatorname{Tors} A$.

For an algebraic closure $F^{\mathrm{alg}}$ of $F$ denote the separable closure of the field $F$ by $F^{\mathrm{sep}}$; let $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ be the absolute Galois group of $F$. Often for a $G_F$-module $M$ we write $H^i(F, M)$ instead of $H^i(G_F, M)$.

For a positive integer $l$ which is prime to characteristic of $F$ (if the latter is non-zero) denote by $\mu_l = \langle \zeta_l \rangle$ the group of $l$ th roots of unity in $F^{\mathrm{sep}}$.

If $l$ is prime to $\operatorname{char}(F)$, for $m \geqslant 0$ denote by $\mathbb{Z}/l(m)$ the $G_F$-module $\mu_l^{\otimes m}$ and put $\mathbb{Z}_l(m) = \varprojlim_r \mathbb{Z}/l^r(m)$; for $m < 0$ put $\mathbb{Z}_l(m) = \mathrm{Hom}(\mathbb{Z}_l, \mathbb{Z}_l(-m))$.

Let $A$ be a commutative ring. The group of invertible elements of $A$ is denoted by $A^*$. Let $B$ be an $A$-algebra. $\Omega^1_{B/A}$ denotes as usual the $B$-module of regular differential forms of $B$ over $A$; $\Omega^n_{B/A} = \wedge^n \Omega^1_{B/A}$. In particular, $\Omega^n_A = \Omega^n_{A/\mathbb{Z}1_A}$ where $1_A$ is the identity element of $A$ with respect to multiplication. For more on differential modules see subsection A1 of the appendix to the section 2 in the first part.

Let $K_n(k) = K_n^M(k)$ be the Milnor $K$-group of a field $k$ (for the definition see subsection 2.0 in the first part).

For a complete discrete valuation field $K$ denote by $\mathcal{O} = \mathcal{O}_K$ its *ring of integers*, by $\mathcal{M} = \mathcal{M}_K$ the *maximal ideal* of $\mathcal{O}$ and by $k = k_K$ its *residue field*. If $k$ is of characteristic $p$, denote by $\mathcal{R}$ the set of *Teichmüller representatives* (or *multiplicative representatives*) in $\mathcal{O}$. For $\theta$ in the maximal perfect subfield of $k$ denote by $[\theta]$ its Teichmüller representative.

For a field $k$ denote by $W(k)$ the ring of Witt vectors (more precisely, Witt $p$-vectors where $p$ is a prime number) over $k$. Denote by $W_r(k)$ the ring of Witt vectors of length $r$ over $k$. If $\operatorname{char}(k) = p$ denote by $\mathbf{F}\colon W(k) \to W(k)$, $\mathbf{F}\colon W_r(k) \to W_r(k)$ the map $(a_0, \dots) \mapsto (a_0^p, \dots)$.

Denote by $v_K$ the surjective discrete valuation $K^* \to \mathbb{Z}$ (it is sometimes called the *normalized discrete valuation* of $K$). Usually $\pi = \pi_K$ denotes a *prime element* of $K$: $v_K(\pi_K) = 1$.

Denote by $K_{\mathrm{ur}}$ the *maximal unramified extension* of $K$. If $k_K$ is finite, denote by $\mathrm{Frob}_K$ the *Frobenius automorphism* of $K_{\mathrm{ur}}/K$.

For a finite extension $L$ of a complete discrete valuation field $K$ $\mathcal{D}_{L/K}$ denotes its different.

If $\operatorname{char}(K) = 0$, $\operatorname{char}(k_K) = p$, then $K$ is called a field of *mixed characteristic*. If $\operatorname{char}(K) = 0 = \operatorname{char}(k_K)$, then $K$ is called a field of *equal characteristic*.

If $k_K$ is perfect, $K$ is called a *local field*.